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Existence and uniqueness of entropy solution to initial boundary value problem for the inviscid Burgers equation

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Abstract

This paper is concerned with the existence and uniqueness of the entropy solution to the initial boundary value problem for the inviscid Burgers equation

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & x > 0 \quad t > 0 \\ u(x,0) = u_0(x) & x \geqslant 0 \\ u(0,t) = 0 & t \geqslant 0. \end{cases}$$

To apply the method of vanishing viscosity to study the existence of the entropy solution, we first introduce the initial boundary value problem for the viscous Burgers equation, and as in Evans (1998 *Partial Differential Equations* (Providence, RI: American Mathematical Society) and Hopf (1950 *Commun. Pure Appl. Math.* **3** 201–30), give the formula of the corresponding viscosity solutions by Hopf–Cole transformation. Secondly, we prove the convergence of the viscosity solution sequences and verify that the limiting function is an entropy solution. Finally, we give an example to show how our main result can be applied to solve the initial boundary value problem for the Burgers equation.

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1. Introduction and the main result

In this paper, we mainly consider the existence and uniqueness of the entropy solution to the initial boundary value problem for the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad x > 0 \quad t > 0 \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x) \qquad x \geqslant 0 \tag{1.2}$$

and boundary condition

$$u(0,t) = 0 \qquad t \geqslant 0. \tag{1.3}$$

It is well known that the existence of the weak solution to the Cauchy problem for the Burgers equation was first studied by Hopf by using the method of vanishing viscosity (cf [7]). He applied the Hopf–Cole transformation to give the formula of the viscosity solutions, and proved the convergence of the corresponding viscosity solution sequences. At time, the weak solution was obtained by the limiting function. The uniqueness of the entropy solution was studied by Oleinik (cf [12]). For the Cauchy problem for general scalar hyperbolic conservation law, the existence of the weak solutions was studied by many authors (cf Tartar [17], Schonbek [14], Chen and Lu [4], Oleinik [12], Rozdestvenskii and Janenko [13], Dafermos [5], also see [6, 16]). Kruzkov in [8] proved the existence and uniqueness of the solution to the Cauchy problem for scalar hyperbolic conservation laws with several space variables. Recently, Bressan, Liu and Yang [2, 3, 11] proved the L^1-stability of the weak solution to the Cauchy problem for the hyperbolic conservation laws.

In this paper, our interest is to study the existence and uniqueness of the entropy solution to the initial boundary value problem (1.1)–(1.3) (cf [9]). It is difficult to study the initial boundary value problem due to the boundary layer. To overcome the difficulties caused by the boundary layer, we need make some technical constriction to initial data (cf (1.7)), which will be used in the proof of (2.20) later.

At first, we give the definition of the entropy solution.

Definition 1.1. *We say a function $u \in L^\infty((0,\infty)\times(0,\infty))$ is an entropy solution to the initial boundary value problem (1.1)–(1.3) provided that u satisfies the following two conditions:*

(i) (integral equation)

$$\int_0^\infty \int_0^\infty \left(u\varphi_t + \frac{u^2}{2}\varphi_x \right) \mathrm{d}x\,\mathrm{d}t + \int_0^\infty u_0\varphi\,\mathrm{d}x|_{t=0} = 0 \tag{1.4}$$

for all test functions $\varphi(x,t) \in C_0^\infty([0,\infty)\times[0,\infty))$.

(ii) (entropy condition)

$$u(x+z,t) - u(x,t) \leqslant C\left(1 + \frac{1}{t}\right) z \tag{1.5}$$

for some constant $C \geqslant 0$ and almost all $x, z \in \mathbb{R}^+$, $t > 0$.

Under the above definition, our main result is stated as follows.

Theorem 1.2. *Suppose*

$$u_0 \in L^\infty([0,\infty)) \tag{1.6}$$

and

$$h(x) := \int_0^x u_0(y)\,\mathrm{d}y \geqslant 0 \tag{1.7}$$

for all $x \geqslant 0$, then there exists a unique entropy solution to the initial boundary value problem (1.1)–(1.3).

The proof of theorem 1.2 is completed by three steps: in step 1, we add an artificial viscosity term $\varepsilon u^\varepsilon_{xx}$ to the right-hand side of (1.1) to get the viscous Burgers equation.

Then a formula on viscosity solution sequences $\{u^\varepsilon(x,t)\}$ is obtained by the Hopf–Cole transformation; in step 2, we consider the limiting behaviour of $u^\varepsilon(x,t)$ as $\varepsilon \to 0$ and show that the limiting function is an entropy solution to (1.1)–(1.3); and in step 3, we prove the uniqueness of the entropy solution.

2. The proof of the main result

We will study the solution to the initial boundary value problem (1.1)–(1.3) by using the method of vanishing viscosity. As in [7], we consider the initial boundary value problem for the viscous Burgers equation

$$\begin{cases} u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon & x > 0 \quad t > 0 \\ u^\varepsilon(x,0) = u_0(x) & x \geqslant 0 \\ u^\varepsilon(0,t) = 0 & t \geqslant 0. \end{cases} \tag{2.1}$$

Lemma 2.1. *The initial boundary value problem (2.1) has a solution for any small parameter $\varepsilon > 0$*

$$u^\varepsilon(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t}\, \mathrm{e}^{-\frac{k(x,y,t)}{2\varepsilon}}\, \mathrm{d}y}{\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{k(x,y,t)}{2\varepsilon}}\, \mathrm{d}y} \qquad x > 0 \quad t > 0 \tag{2.2}$$

where

$$k(x,y,t) = \frac{(x-y)^2}{2t} + \tilde{h}(y) \qquad x \geqslant 0 \quad t \geqslant 0 \quad y \in \mathbb{R} \tag{2.3}$$

and

$$\tilde{h}(y) = \begin{cases} h(y) & \text{if } y \geqslant 0 \\ h(-y) & \text{if } y < 0. \end{cases} \tag{2.4}$$

Proof. As in [6, 7, 13], let

$$w^\varepsilon(x,t) = \int_0^x u^\varepsilon(y,t)\, \mathrm{d}y \qquad x \geqslant 0 \quad t \geqslant 0. \tag{2.5}$$

Then (2.1) is rewritten as follows:

$$\begin{cases} w_t^\varepsilon - \varepsilon w_{xx}^\varepsilon + \frac{1}{2}\left(w_x^\varepsilon\right)^2 = 0 & x > 0 \quad t > 0 \\ w^\varepsilon(x,0) = h(x) & x \geqslant 0 \\ w_x^\varepsilon(0,t) = 0 & t \geqslant 0. \end{cases} \tag{2.6}$$

We introduce the Hopf–Cole transformation

$$v^\varepsilon(x,t) = \mathrm{e}^{-\frac{w^\varepsilon(x,t)}{2\varepsilon}} \qquad x \geqslant 0 \quad t \geqslant 0 \tag{2.7}$$

then $v^\varepsilon(x,t)$ solves the following initial boundary value problem for the heat equation (with conductivity ε):

$$\begin{cases} v_t^\varepsilon - \varepsilon v_{xx}^\varepsilon = 0 & x > 0 \quad t > 0 \\ v^\varepsilon(x,0) = \mathrm{e}^{-\frac{h(x)}{2\varepsilon}} & x \geqslant 0 \\ v_x^\varepsilon(0,t) = 0 & t \geqslant 0 \end{cases} \tag{2.8}$$

the unique bounded solution of which is

$$v^\varepsilon(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_0^\infty \left(\mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}} + \mathrm{e}^{-\frac{(x+y)^2}{4\varepsilon t}}\right) \mathrm{e}^{-\frac{h(y)}{2\varepsilon}}\, \mathrm{d}y. \tag{2.9}$$

From (2.5) and (2.7), we have

$$u^\varepsilon(x,t) = \frac{\partial}{\partial x} w^\varepsilon(x,t) = -2\varepsilon \frac{v_x^\varepsilon(x,t)}{v^\varepsilon(x,t)}. \tag{2.10}$$

Furthermore, by (2.9), (2.3) and (2.4), (2.10) can be rewritten as

$$\begin{aligned}
u^\varepsilon(x,t) &= \frac{\int_0^\infty \left(\frac{x-y}{t}\,\mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}} + \frac{x+y}{t}\,\mathrm{e}^{-\frac{(x+y)^2}{4\varepsilon t}}\right)\mathrm{e}^{-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y}{\int_0^\infty \left(\mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}} + \mathrm{e}^{-\frac{(x+y)^2}{4\varepsilon t}}\right)\mathrm{e}^{-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y} \\
&= \frac{\int_0^\infty \frac{x-y}{t}\,\mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y + \int_0^\infty \frac{x+y}{t}\,\mathrm{e}^{-\frac{(x+y)^2}{4\varepsilon t}-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y}{\int_0^\infty \mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y + \int_0^\infty \mathrm{e}^{-\frac{(x+y)^2}{4\varepsilon t}-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y} \\
&= \frac{\int_0^\infty \frac{x-y}{t}\,\mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y + \int_{-\infty}^0 \frac{x-y}{t}\,\mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}-\frac{h(-y)}{2\varepsilon}}\,\mathrm{d}y}{\int_0^\infty \mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}-\frac{h(y)}{2\varepsilon}}\,\mathrm{d}y + \int_{-\infty}^0 \mathrm{e}^{-\frac{(x-y)^2}{4\varepsilon t}-\frac{h(-y)}{2\varepsilon}}\,\mathrm{d}y} \\
&= \frac{\int_{-\infty}^\infty \frac{x-y}{t}\,\mathrm{e}^{-\frac{k(x,y,t)}{2\varepsilon}}\,\mathrm{d}y}{\int_{-\infty}^\infty \mathrm{e}^{-\frac{k(x,y,t)}{2\varepsilon}}\,\mathrm{d}y}.
\end{aligned}$$

This completes the proof of lemma 2.1. □

Now we consider the limiting behaviour of the viscosity solution sequences $\{u^\varepsilon(x,t)\}$ given by (2.2) as $\varepsilon \to 0$. For this purpose, we first state two lemmas, whose proof can be found in [6].

Lemma 2.2. *Assume $u_0 \in L^\infty([0,\infty))$, then*

(i) for each time $t > 0$, there exists a unique point $y(x,t)$ for all but at most countably many values of $x \in [0,\infty)$ such that

$$\min_{y\in\mathbb{R}} k(x,y,t) = k(x,y(x,t),t) \tag{2.11}$$

(ii) the mapping $x \mapsto y(x,t)$ is nondecreasing.

Lemma 2.3. *Suppose that $l, m : \mathbb{R} \to \mathbb{R}$ are continuous functions, satisfying*

(i) l grows at most linearly and m grows at least quadratically;
(ii) there exists a unique point $y_0 \in \mathbb{R}$ such that

$$m(y_0) = \min_{y\in\mathbb{R}} m(y).$$

Then

$$\lim_{\varepsilon\to 0} \frac{\int_{-\infty}^\infty l(y)\,\mathrm{e}^{-\frac{m(y)}{2\varepsilon}}\,\mathrm{d}y}{\int_{-\infty}^\infty \mathrm{e}^{-\frac{m(y)}{2\varepsilon}}\,\mathrm{d}y} = l(y_0). \tag{2.12}$$

Owing then to lemmas 2.2 and 2.3, we easily find the limiting behaviour of the solution $u^\varepsilon(x,t)$ to the initial boundary value problem (2.1) for the viscous Burgers equation. That is, we have

Theorem 2.4. *For almost all $x > 0$ and $t > 0$,*

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = \lim_{\varepsilon\to 0} \frac{\int_{-\infty}^\infty \frac{x-y}{t}\,\mathrm{e}^{-\frac{k(x,y,t)}{2\varepsilon}}\,\mathrm{d}y}{\int_{-\infty}^\infty \mathrm{e}^{-\frac{k(x,y,t)}{2\varepsilon}}\,\mathrm{d}y} = \frac{x - y(x,t)}{t} \tag{2.13}$$

where $y(x,t)$ is defined by (2.11).

Next we will verify that $u(x,t) = \frac{x-y(x,t)}{t}$ is an entropy solution to the initial boundary value problem (1.1)–(1.3). For doing this, we give the following lemmas:

Lemma 2.5. *The initial boundary value problem for the Hamilton–Jacobi equation*

$$\begin{cases} w_t + \frac{1}{2}w_x^2 = 0 & x > 0 \quad t > 0 \\ w(x,0) = h(x) & x \geqslant 0 \\ w_x(0,t) = 0 & t \geqslant 0 \end{cases} \tag{2.14}$$

has a unique entropy solution

$$w(x,t) = \min_{y\in\mathbb{R}} k(x,y,t) \qquad x \geqslant 0 \quad t \geqslant 0 \tag{2.15}$$

that makes the following results hold:

(i) $w(x,t)$ *is Lipschitz continuous in* $[0,\infty)\times[0,\infty)$,
(ii) $w_t + \frac{1}{2}w_x^2 = 0$, *almost everywhere* $(x,t) \in (0,\infty)\times(0,\infty)$,
(iii) $w(x,0) = h(x)$, $x \geqslant 0$ *and*
(iv) $w_x(0,t) = 0$, $t \geqslant 0$.

Proof. Define

$$\tilde{h}(x) = \begin{cases} h(x) & x \geqslant 0 \\ h(-x) & x < 0. \end{cases}$$

Consider the Cauchy problem for the Hamilton–Jacobi equation

$$\begin{cases} \tilde{w}_t + \frac{1}{2}\tilde{w}_x^2 = 0 & x \in \mathbb{R} \quad t > 0 \\ \tilde{w}(x,0) = \tilde{h}(x) & x \in \mathbb{R}. \end{cases} \tag{2.16}$$

By using the results in [6], there exists a unique entropy solution $\tilde{w}(x,t)$ to (2.16) satisfying

(α) $\tilde{w}(x,t)$ is Lipschitz continuous in $\mathbb{R}\times\mathbb{R}^+$,
(β) $\tilde{w}(x,t)$ satisfies the equation $\tilde{w}_t + \frac{1}{2}\tilde{w}_x^2 = 0$ for almost everywhere $x \in \mathbb{R}$, $t \in \mathbb{R}^+$ and
(γ) $\tilde{w}(x,0) = \tilde{h}(x)$, $x \in \mathbb{R}$.

Now we define for $x \geqslant 0$, $t \geqslant 0$

$$w(x,t) = \tilde{w}(x,t)\mid_{x\geqslant 0}.$$

Then $w(x,t)$ satisfies (i), (ii) and (iii) in lemma 2.5. Now we verify that $w(x,t)$ satisfies (iv).

In fact, it is easy to verify that $\tilde{w}_1(x,t) = \tilde{w}(-x,t)$ also is an entropy solution to (2.16). By the uniqueness of the solution to the Cauchy problem for the Hamilton–Jacobi equation, we deduce

$$\tilde{w}(x,t) = \tilde{w}(-x,t) \qquad x \in \mathbb{R} \quad t \in \mathbb{R}^+$$

which implies

$$\tilde{w}_x(0,t) = 0 \qquad t \geqslant 0$$

i.e. $w_x(0,t) = 0$, $t \geqslant 0$.

This proves lemma 2.5. □

Note that $w(x,t) = \min_{y\in\mathbb{R}} k(x,y,t)$ is Lipschitz continuous, then differentiable for almost all (x,t). We define

$$u_1(x,t) := \frac{\partial}{\partial x}w(x,t) = \frac{\partial}{\partial x}\min k(x,y,t) \tag{2.17}$$

for almost all $(x,t) \in [0,\infty)\times[0,\infty)$. Then we have the following lemma. The proof can be found in [6].

Lemma 2.6. *$u_1(x,t)$ defined by (2.17) can be expressed as*

$$u_1(x,t) = \frac{x - y(x,t)}{t}$$

for almost all $(x,t) \in [0,\infty) \times [0,\infty)$, where $y(x,t)$ is defined by (2.11).

Thus

$$u(x,t) = u_1(x,t) = \frac{\partial}{\partial x} w(x,t) = \frac{x - y(x,t)}{t}.$$

Next, we give the existence theorem of the entropy solution to the initial boundary value problem (1.1)–(1.3).

Theorem 2.7 (Existence)**.** *The limiting function*

$$u(x,t) = \frac{x - y(x,t)}{t} \qquad x \geqslant 0 \quad t \geqslant 0$$

which is obtained by $u^{\varepsilon}(x,t)$ (cf (2.13)), is an entropy solution to the initial boundary value problem (1.1)–(1.3). Here $y(x,t)$ is defined by (2.11).

Proof. Set $w(x,t) = \min_{y\in\mathbb{R}} k(x,y,t)$ $(x \geqslant 0, t \geqslant 0)$. Then lemma 2.5 shows that $w(x,t)$ is Lipschitz continuous, differentiable for a.e. (x,t) and solves

$$\begin{cases} w_t + \frac{1}{2} w_x^2 = 0 & \text{a.e. } (x,t) \in [0,\infty) \times [0,\infty) \\ w(x,0) = h(x) & x \geqslant 0 \\ w_x(0,t) = 0 & t \geqslant 0. \end{cases}$$

Choose any test function $\varphi \in C_0^{\infty}([0,\infty) \times [0,\infty))$, then

$$\int_0^{\infty} \int_0^{\infty} \left(w_t + \frac{1}{2} w_x^2 \right) \varphi_x \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{2.18}$$

Note that after two integrations by parts, we have

$$\begin{aligned} \int_0^{\infty} \int_0^{\infty} w_t \varphi_x \, \mathrm{d}x \, \mathrm{d}t &= -\int_0^{\infty} \int_0^{\infty} w \varphi_{tx} \, \mathrm{d}x \, \mathrm{d}t + \int_0^{\infty} w \varphi_x \, \mathrm{d}x \bigg|_{t=0}^{t=\infty} \\ &= -\int_0^{\infty} \int_0^{\infty} w \varphi_{tx} \, \mathrm{d}x \, \mathrm{d}t - \int_0^{\infty} w \varphi_x \, \mathrm{d}x \bigg|_{t=0} \\ &= \int_0^{\infty} \int_0^{\infty} w_x \varphi_t \, \mathrm{d}x \, \mathrm{d}t - \int_0^{\infty} w \varphi_t \, \mathrm{d}t \bigg|_{x=0}^{x=\infty} \\ &\quad - \left(w\varphi|_{x=0}^{x=\infty} - \int_0^{\infty} w_x \varphi \, \mathrm{d}x \right) \bigg|_{t=0} \\ &= \int_0^{\infty} \int_0^{\infty} w_x \varphi_t \, \mathrm{d}x \, \mathrm{d}t + \int_0^{\infty} w_x \varphi \, \mathrm{d}x \bigg|_{t=0} + \int_0^{\infty} w \varphi_t \, \mathrm{d}t \bigg|_{x=0} + w\varphi \bigg|_{x=0}. \end{aligned} \tag{2.19}$$

On the other hand

$$w(0,t) = \min_{y\in\mathbb{R}} k(0,y,t) = \min_{y\in\mathbb{R}} \left\{ \frac{y^2}{2t} + \tilde{h}(y) \right\}.$$

By (1.7) and (2.4) we see $\tilde{h}(y) \geqslant 0$ $(y \in \mathbb{R})$. It follows that

$$\frac{y^2}{2t} + \tilde{h}(y) > 0 \qquad \text{for any} \quad y \neq 0 \quad t > 0$$

and

$$\left(\frac{y^2}{2t} + \tilde{h}(y)\right)\bigg|_{y=0} = \tilde{h}(0) = 0.$$

Thus

$$w(0,t) = 0 \qquad t > 0 \tag{2.20}$$

which implies from (2.19)

$$\int_0^\infty \int_0^\infty w_t \varphi_x \,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_0^\infty w_x \varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty w_x \varphi \,\mathrm{d}x \bigg|_{t=0}. \tag{2.21}$$

Substituting (2.21) into (2.18), we have

$$\int_0^\infty \int_0^\infty \left(w_x \varphi_t + \frac{1}{2} w_x^2 \varphi_x\right) \mathrm{d}x\,\mathrm{d}t + \int_0^\infty w_x \varphi \,\mathrm{d}x \bigg|_{t=0} = 0. \tag{2.22}$$

Since $u(x,t) = \frac{x - y(x,t)}{t} = w_x(x,t)$ for almost all $(x,t) \in (0,\infty) \times (0,\infty)$, then (2.22) becomes

$$\int_0^\infty \int_0^\infty \left(u \varphi_t + \frac{1}{2} u^2 \varphi_x\right) \mathrm{d}x\,\mathrm{d}t + \int_0^\infty u_0 \varphi \,\mathrm{d}x \bigg|_{t=0} = 0$$

which show that $u(x,t) = \frac{x - y(x,t)}{t}$ is the integral solution to the initial boundary value problem (1.1)–(1.3).

Finally, we prove that the entropy condition (1.5) holds. In fact, according to [6], we have

$$u(x+z,t) - u(x,t) \leqslant \frac{C}{t} z$$

for some constant $C \geqslant 0$ and $x, z \in \mathbb{R}^+$, $t > 0$. This completes the proof of theorem 2.7. □

Theorem 2.8 (Uniqueness)**.** *The entropy solution defined by definition 1.1 is unique.*

Proof. Similar to lemma 2.5, we easily prove that if $u(x,t)(x \geqslant 0, t \geqslant 0)$ is a solution to (1.1), then $-u(-x,t)$ $(x < 0, t > 0)$ is also. Noting that $u(0,t) = 0$ $(t \geqslant 0)$, we extend u, u_0 to all of $\mathbb{R}$ by odd reflection, that is

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & x \geqslant 0 \quad t \geqslant 0 \\ -u(-x,t) & x < 0 \quad t \geqslant 0 \end{cases}$$

and

$$\tilde{u}_0(x) := \begin{cases} u_0(x) & x \geqslant 0 \\ -u_0(-x) & x < 0. \end{cases}$$

Then $\tilde{u}(x,t)$ solves the following Cauchy problem:

$$\begin{cases} \tilde{u}_t + \tilde{u}\tilde{u}_x = 0 & x \in \mathbb{R} \quad t > 0 \\ \tilde{u}(x,0) = \tilde{u}_0(x) & x \in \mathbb{R}. \end{cases} \tag{2.23}$$

Evans [6] shows us that there exists—up to a set of measure zero—at most one entropy solution to the Cauchy problem (2.23). So there exists at most one entropy solution to the initial boundary value problem (1.1)–(1.3). This completes the proof of theorem 2.8. □

Theorems 2.7 and 2.8 imply theorem 1.2.

3. An example

In this section, we give an example to show how our main result can be applied to solve the initial boundary value problem for the Burgers equation.

Let the initial data $u_0(x)$ $(x \geqslant 0)$ in (1.2) satisfy the following three conditions:

(i) $\lim_{x\to 0+} u_0(x) = u_0 > 0$,
(ii) $\lim_{x\to\infty} u_0(x) = u_+$ and
(iii) $u_0(x)(x \geqslant 0)$ is a nondecreasing, continuous function in $[0, \infty)$.

Then, by the method of characteristics, it is easy to get the unique entropy solution to the initial boundary value problem (1.1)–(1.3), which has the following form for $(x, t) \in [0, \infty) \times [0, \infty)$:

$$u(x,t) := \begin{cases} \dfrac{x}{t} & x \leqslant u_0 t \\ u_0(y(x,t)) & x > u_0 t \end{cases} \tag{3.1}$$

where $y = y(x, t)$ denotes the unique solution to the equation

$$t u_0(y) = x - y \qquad x > u_0 t. \tag{3.2}$$

Now we will show that the unique entropy solution stated by theorem 1.2 also has the form (3.1), that is the function $u(x, t)$ defined by (3.1) can be restated as

$$u(x,t) = \frac{x - y(x,t)}{t}$$

where $y(x, t)$ is defined by (2.11).

Choose any $x_0 > 0, t_0 > 0$. We have to find $\min_{y\in\mathbb{R}} k(x_0, y, t_0)$ and the optimizer $y_0 = y(x_0, t_0)$ when the minimum is taken.

Let

$$k(y) := k(x_0, y, t_0) = \frac{(x_0 - y)^2}{2t_0} + \tilde{h}(y) \qquad y \in \mathbb{R}. \tag{3.3}$$

Since $\tilde{h}(y)$ $(y \in \mathbb{R})$ is an even function and is increasing in $[0, \infty)$. So $k(y)$ is decreasing in $(-\infty, 0]$ and increasing in $[x_0, \infty)$. Thus we have that

$$\min_{y\in\mathbb{R}} k(y) = \min_{y\in[0,x_0]} k(y).$$

On the other hand, note that

(i) when $y = 0$, $k(0) = \frac{x_0^2}{2t_0} = \int_0^{x_0} \frac{y}{t_0}\,\mathrm{d}y$
(ii) when $y = x_0$, $k(x_0) = \tilde{h}(x_0) = \int_0^{x_0} u_0(y)\,\mathrm{d}y$
(iii) when $0 < y < x_0$, $k_y(y) = \frac{y - x_0}{t_0} + u_0(y)$.

Then we can get the optimizer y_0.

Case 1. When $x_0 \leqslant u_0 t_0$, the equation $k_y(y) = 0$ has no solution in $(0, x_0)$ due to the monotony of $u_0(x)$ $(x \geqslant 0)$, which shows the optimizer y_0 cannot be obtained in $(0, x_0)$. Noting that $k(0) < k(x_0)$, we have $y_0 = y(x_0, t_0) = 0$. Thus

$$u(x_0, t_0) = \frac{x_0 - y(x_0, t_0)}{t_0} = \frac{x_0}{t_0}. \tag{3.4}$$

Case 2. When $x_0 > u_0 t_0$, the equation $k_y(y) = 0$, that is

$$t_0 u_0(y) = x_0 - y$$

has a unique solution due to the monotony and continuity of $u_0(x)$ $(x \geqslant 0)$, which shows the optimizer y_0 can be uniquely obtained in $(0, x_0)$. So we have

$$u(x_0, t_0) = \frac{x_0 - y(x_0, t_0)}{t_0} = u_0(y_0) = u_0(y(x_0, t_0)). \tag{3.5}$$

These arguments show that the solution found by the method of characteristics is just one generated by theorem 1.2.

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